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On local Calabi–Yau supermanifolds and their mirrors

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Abstract

We use local mirror symmetry to study a class of local Calabi–Yau supermanifolds with bosonic sub-variety V_b having a vanishing first Chern class. Solving the usual super-CY condition, requiring the equality of the total $U(1)$ gauge charges of bosons Φ_b and the ghost-like fields Ψ_f one $\sum_b q_b = \sum_f Q_f$, as $\sum_b q_b = 0$ and $\sum_f Q_f = 0$, several examples are studied and explicit results are given for local A_r supergeometries. A comment on purely fermionic super-CY manifolds corresponding to the special case where $q_b = 0, \forall b$ and $\sum_f Q_f = 0$ is also made.

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1. Introduction

Mirror symmetry has played a crucial role in superstring dualities. It provides a map between Calabi–Yau (CY) manifolds used in the compactification of 10D superstring models and topological string theory. In particular, the topological A- and B-models are connected by mirror symmetry, as discussed below. However, it has been realized, though, that rigid CY manifolds can have mirror manifolds which are not themselves CY geometries. An intriguing remedy is the introduction of CY *supermanifolds* in these considerations [1, 2]. It has thus been suggested that mirror symmetry is between supermanifolds and manifolds alike, and not just between bosonic manifolds.

On the other hand, it has been found that there is a correspondence between the moduli space of holomorphic Chern–Simons theory on the CY supermanifold $\mathbf{CP}^{3|4}$ and the self-dual, four-dimensional $N = 4$ Yang–Mills theory [3, 4]. This may also be related to the B-model of an open topological string theory having $\mathbf{CP}^{3|4}$ as target space. Partly based on this work, CY supermanifolds and their mirrors have subsequently attracted a great deal of attention [5–16]. It has been found, for instance, that an A-model defined on the CY supermanifold $\mathbf{CP}^{3|4}$ is a mirror of a B-model on a quadric hypersurface in $\mathbf{CP}^{3|3} \times \mathbf{CP}^{3|3}$, provided the Kähler parameter of $\mathbf{CP}^{3|4}$ approaches infinity [5, 6].

Following this observation, an effort has been devoted to go beyond these particular geometries. A special interest has been given to construct the mirror of Calabi–Yau supermanifolds whose bosonic parts are compact toric varieties [17]. One of the objectives of the present work is to extend the result of [17] by considering local Calabi–Yau manifolds which have been used in type II superstring compactifications in the presence of D-branes. In particular, we discuss the mirror symmetry of the topological A-model on supermanifolds whose bosonic part is a local CY variety. The corresponding theory is a supersymmetric $U(1)^p$ linear sigma model with $(n + p)$ chiral superfields with charge q_i^a and $2p$ fermionic superfields with charge given by Q_α^a which is a $p \times 2p$ matrix. These charges satisfy the superlocal CY (SLCY) condition $\sum_{i=1}^{p+n} q_i^a - \sum_{\alpha=1}^{2p} Q_\alpha^a = 0$ requiring equality between the total charge of bosons and the ghost-like fields.

In this paper, we shall focus on the mirror supergeometry obtained by first choosing a special form of the full spectrum of $U(1)^p$ gauge charges and integrating out some fermionic fields in the topological B-model. In this way, the mirror B-models will still have some fermionic directions. Our interest will be on the mirror of ADE supergeometries and mainly on the fermionic extension of the ordinary A_r class. First, we study the case of A_1 supergeometry, which is found to be closely related to the equation of the bosonic case in agreement with the analysis using Landau–Ginzburg (LG) models. Ordinary A_1 geometry is recovered by cancelling the fermionic directions. Then we work out the mirror of a class of A_r local super-CY manifold extending the A_1 supergeometry. Finally, we discuss the mirror symmetry of local higher dimensional super-CY geometries. In particular, we specialize on the mirror symmetry of the topological A-model using a fermionic extension of a line bundle over $\mathbf{CP}^n$.

The organization of this paper is as follows. In section 2, we review mirror symmetry of local super-CY manifolds. In section 3, we study the mirror of ADE supergeometries by exhibiting the method on the ordinary A_r series. In section 4, we consider mirror supergeometries beyond ADE and in section 5, we give a conclusion.

2. Mirror symmetry of local super-CY manifolds

In this section, we review mirror symmetry for local (bosonic) CY manifolds [18, 19], then we give the extension to the supercase.

2.1. Bosonic CY

To begin with, let us consider a two-dimensional $\mathcal{N} = 2$ supersymmetric linear sigma model described in terms of $n+p$ chiral superfields Φ_i with charge q_i^a, $i = 1, \ldots, n+p$, $a = 1, \ldots, p$ under $U(1)^{\otimes p}$ gauge symmetry [20]. The geometry of the topological A-model can be analysed by solving the D-term potential ($D^a = 0$) of the $\mathcal{N} = 2$ linear sigma model; that is

$$\sum_{i=1}^{n+p} q_i^a |\phi_i|^2 = r^a, \qquad a = 1, \ldots, p, \tag{2.1}$$

where the r^a's are FI coupling parameters and the ϕ_i's are the leading scalar fields of the chiral superfield Φ_i. Dividing by $U(1)^{\otimes p}$ gauge symmetry, one gets an n-dimensional toric variety[4]

$$\mathbf{V}^n = \frac{\mathbb{C}^{n+p} \setminus S}{\mathbb{C}^{*p}}, \tag{2.2}$$

where the p copies of $\mathbb{C}^*$ actions indexed by $a = 1, \ldots, p$, are given by

$$\mathbb{C}^{*p} : \phi_i \to \lambda^{q_i^a} \phi_i, \qquad i = 1, \ldots, n+p, \tag{2.3}$$

with λ a non-zero complex number. The requirement for $\mathbf{V}^n$ to be a local CY manifold is to impose the condition

$$\sum_{i=1}^{n+p} q_i^a = 0. \tag{2.4}$$

On the supersymmetric field theoretic level, this relation implies that the underlying linear sigma model flows in infrared to a conformal field theory.

Following [18–22], the mirror B-model is an LG model with periodic fields $\{Y_i\}$ dual to $\{\Phi_i\}$ and connected as

$$\mathrm{Re}(Y_i) = |\Phi_i|^2, \qquad i = 1, \ldots, n+p, \tag{2.5}$$

where $\mathrm{Re}(Y_i)$ denotes the real part of Y_i. Under mirror transformation, equation (2.1) is mapped to

$$\sum_i q_i^a Y_i = t^a, \qquad a = 1, \ldots, p, \tag{2.6}$$

with $r^a = \mathrm{Re}(t^a)$. Moreover, the LG superpotential of the topological B-model reads as

$$W(Y_1, \ldots, Y_{n+p}) = \sum_{i=1}^{n+p} \mathrm{e}^{-Y_i}. \tag{2.7}$$

For convenience, it is useful to use the following field redefinitions:

$$\hat{y}_i = \mathrm{e}^{-Y_i}, \qquad i = 1, \ldots, n+p, \tag{2.8}$$

Then the superpotential $W = W(\hat{y}_1, \ldots, \hat{y}_{n+p})$ reads as

$$W = \sum_{i=1}^{n+p} \hat{y}_i, \tag{2.9}$$

and so equation (2.6) translates into the following projective hypersurface:

$$\prod_{i=1}^{n+p} \hat{y}_i^{q_i^a} = \mathrm{e}^{-t_a}, \qquad a = 1, \ldots, p, \tag{2.10}$$

with the manifest projective symmetry $\hat{y}_i \to \lambda \hat{y}_i$ following from the CY condition (2.4). The solution of the constraint equation (2.10) and projective symmetry defines a $(n+p) - p - 1 - 1 = n - 2$ dimensional toric manifold given by a holomorphic hypersurface in $\mathbb{C}^{n-1}$:

$$F(y_1, \ldots, y_{n-1}) = 0. \tag{2.11}$$

To recover the right dimension of the original manifold, that is a complex dimension n local CY manifold, we generally use an ad hoc trick which consists of adding by hand two extra

[4] Note that this geometry can be represented by a toric diagram $\Delta(\mathbf{V}^n)$ spanned by $k = n + p$ vertices v_i in a $\mathbb{Z}^n$ lattice satisfying $\sum_{i=1}^{n+p} q_i^a v_i = 0, \quad a = 1, \ldots, p$.

holomorphic variables u and v combined in a quadratic form uv and modifying the previous equation as

$$F(y_1, \dots, y_{n-1}) = uv. \tag{2.12}$$

The main objective in what follows is to extend this analysis to a linear A-model with fermionic (ghosts) fields and study the resulting mirror B-model. Besides the generalization of the above results to local super-CY manifolds, one of the results following from this fermionic extension is the re-derivation of equation (2.12) without the need of adding by hand of the term uv of right-hand aide. As we will show later, the new manifold is given by a hypersurface of type

$$G(y_1, \dots, y_{n-1}) = \chi\eta, \tag{2.13}$$

where, instead of u and v variables, we have now the variables χ and η which are ghost-like fields. As we will see, this relation defines an even complex n dimension hypersurface of the complex superspace $\mathbb{C}^{(n-1)|2}$. This geometry may then be viewed as an alternative elevation of (2.11). The standard elevation equation (2.12) is given by the purely bosonic hypersurface in $\mathbb{C}^{(n+1)|0}$.

2.2. *Mirror of local super-CY*

Here, we want to study the mirror of the fermionic extension of the topological A-model on local toric CY manifolds discussed in the previous subsection. Actually, this may be viewed as an extension of the paper [17] which has dealt with the case of compact bosonic toric manifolds. Important examples of that work have been projective spaces and products thereof.

2.2.1. Extended A-model. Roughly, the extension corresponds to adding, to the usual bosonic superfield Φ_j, a set of f-fermionic chiral superfields Ψ_α with Q_f^a charge under $U(1)^{\otimes p}$ gauge symmetry. We then have

$$\begin{aligned} \Phi_j &\to \mathrm{e}^{\mathrm{i}\sum_a \vartheta_a q_j^a}\Phi_j, && j = 1, \dots, n+p, \\ \Psi_\alpha &\to \mathrm{e}^{\mathrm{i}\sum_a \vartheta_a Q_\alpha^a}\Psi_\alpha, && \alpha = 1, \dots, f, \end{aligned} \tag{2.14}$$

with the same transformations for the leading component fields ϕ_j and ψ_α respectively and where ϑ_a's are the gauge group parameters. The full spectrum of $U(1)^{\otimes p}$ charge vectors $q'^a = (q^a|Q^a)$ thus takes the form

$$(q^a|Q^a) = \left(q_1^a, \dots, q_{p+n}^a \,\middle|\, Q_1^a, \dots, Q_f^a\right), \qquad a = 1, \dots, p. \tag{2.15}$$

The extended D^a-term equations resulting from the above generalized A-model are given by minimizing the Kahler potential of the $2\mathrm{D}\mathcal{N} = 2$ generalized superfield action:

$$\mathcal{S}_{\mathcal{N}=2} = \int \mathrm{d}^2\sigma\, \mathrm{d}^4\theta \mathcal{K} + \left(\int \mathrm{d}^2\sigma\, \mathrm{d}^2\theta \mathrm{W} + \mathrm{cc}\right), \tag{2.16}$$

with respect to the gauge superfields V_a. In the above relation, $\mathcal{K}$ is the usual gauge invariant Kahler term and W is a chiral superpotential with the superfield dependence as,

$$\begin{aligned} \mathcal{K} &= \mathcal{K}\left[\Phi_1, \dots, \Phi_{n+p}^+; \Psi_1, \dots, \Psi_f^+; V_1, \dots, V_p\right], \\ \mathrm{W} &= \mathrm{W}[\Phi_1, \dots, \Phi_{n+p}; \Psi_1, \dots, \Psi_f], \end{aligned} \tag{2.17}$$

as well as coupling constant moduli which have not been specified. Using the explicit expression of $\mathcal{K}$ and putting back into

$$\mathrm{D}^a = \left.\frac{\partial \mathcal{K}}{\partial V_a}\right|_{\theta=0} = 0, \tag{2.18}$$

we get the following D^a-term equations:

$$\sum_{i=1}^{p+n} q_i^a |\phi_i|^2 + \sum_{\alpha=1}^{f} Q_\alpha^a |\psi_\alpha|^2 = \mathrm{Re}(t^a), \qquad a = 1, \ldots, p, \tag{2.19}$$

where $\mathrm{Re}(t^a)$ stands for the FI coupling constant. Strictly speaking, this is an even hypersurface embedded in the complex supermanifold $\mathbb{C}^{p+n|f}$ with dimension $(p+n|f)$. Therefore, the space of vacua of the above generalized supersymmetric action is a toric supermanifold $\mathcal{V}^n$ obtained by dividing $\mathbb{C}^{p+n|f}$ by $U(1)^{\otimes p}$ gauge symmetry group in the same logic as in equation (2.2). Thus we have

$$\mathcal{V}^n = \frac{\mathbb{C}^{p+n|f}\backslash \mathrm{S}}{\mathbb{C}^{*p}}. \tag{2.20}$$

With this relation at hand, one can go ahead and try to develop the toric supergeometry of these local supermanifolds by mimicking the standard toric geometry analysis of toric varieties. We shall not do this here; what we will do rather is to study some specific examples with a direct link to type II superstring theory compactifications. The first class of these examples concerns specific fermionic extensions of ADE geometries. The local super-CY (LSCY) condition reads as follows:

$$\sum_{i=1}^{p+n} q_i^a - \sum_{\alpha=1}^{f} Q_\alpha^a = 0. \tag{2.21}$$

This constraint equation is required by the invariance of holomorphic measure $\Omega^{(p+n|f)}$ of the complex superspace $\mathbb{C}^{p+n|f}$,

$$\Omega^{(p+n|f)} = \left(\textstyle\prod_{i=1}^{n+p} \mathrm{d}\phi_i\right)\left(\textstyle\prod_{\alpha=1}^{f} \mathrm{d}\psi_\alpha\right), \tag{2.22}$$

under $\mathbb{C}^{*p}$ toric symmetry. In what follows, we shall fix our attention on those local CY supermanifolds $\mathcal{V}^n$ obeying the following special solution:

$$\sum_{i=1}^{p+n} q_i^a = 0, \qquad \sum_{\alpha=1}^{f} Q_\alpha^a = 0, \qquad a = 1, \ldots, p. \tag{2.23}$$

In this particular class of solutions of equation (2.21), we have taken the bosonic sub-variety as a local CY manifold. This is the case for bosonic sub-varieties given by the fermionic extensions of ADE geometries we are interested in here. It is also remarkable that fermionic directions obey as well a CY condition for the bosonic manifold. In section 5, we will make a comment on this issue.

2.2.2. Extended B-model The dual extended B-model geometries were derived in [6] by using the T-duality in the linear sigma model describing the A-model. The result of [6] is as follows. Consider a U(1) gauge linear sigma model with fermionic fields Φ_α. As in the bosonic model, a fermionic superfield Φ is dualized by a triplet (X, η, χ). These fields are related by

$$\mathrm{Re}(X) = -|\Psi|^2. \tag{2.24}$$

The ghost fields (η, χ) have been added to guarantee the same superdimension in the mirror geometry. This supersymmetric extension gives a contribution, in the superpotential for the mirror theory, of the form

$$\mathrm{e}^{-X}(1 + \eta\chi). \tag{2.25}$$

Using this technique, let us review briefly the mirror of CY supermanifold $\mathbf{CP}^{3|4}$ [6]. The manifold has a linear sigma model description in terms of 4 bosonic and 4 fermionic fields with charge $(1, 1, 1, 1|1, 1, 1, 1)$. In this way, equation (2.19) reduces to

$$\sum_{i=1}^{4} |\phi_i|^2 + \sum_{\alpha=1}^{4} |\psi_\alpha|^2 = \mathrm{Re}(t). \tag{2.26}$$

The mirror theory is given in terms of the following patch integral:

$$\mathcal{Z} = \int \prod_{i,\alpha=1}^{4} \mathrm{d}Y_i\, \mathrm{d}X_\alpha\, \mathrm{d}\eta_\alpha\, \mathrm{d}\chi_\alpha \delta\left(\sum_{i=1}^{4} Y_i - \sum_{\alpha=1}^{4} X_\alpha - t\right) \exp\left(\sum_{i=1}^{4} \mathrm{e}^{-Y_i} + \sum_{\alpha=1}^{4} \mathrm{e}^{-X_\alpha}(1 + \eta_\alpha\chi_\alpha)\right) \tag{2.27}$$

After field redefinitions ($x_i = \mathrm{e}^{-X_i}$, $Y_i = \mathrm{e}^{-Y_i}$), and integrating the delta function, the mirror geometry takes the form

$$\sum_{i=1}^{3} x_i y_i + x_i + 1 + \mathrm{e}^t y_1 y_2 y_3 + \eta_i \chi_i = 0. \tag{2.28}$$

If we take the limit $t \to -\infty$, one gets a quadric hypersurface in $\mathbf{CP}^{3|3} \times \mathbf{CP}^{3|3}$.

In what follows, we follow the same line to study a class of supermanifolds with local toric CY geometries in the bosonic part involving more than one U(1) gauge symmetries (2.19). Under T-duality, the bosonic superfield Φ_i of the linear supertoric sigma model is replaced by a dual superfield Y_i as before, while the fermionic superfield Ψ_α is dualized by the triplet $(X_\alpha, \eta_\alpha, \chi_\alpha)$ [6]. The bosonic superfields X_α are related to Ψ_α as

$$\mathrm{Re}(X_\alpha) = -|\Psi_\alpha|^2, \qquad \alpha = 1, \ldots, f, \tag{2.29}$$

and the accompanying pair of chiral superfields $\{\eta_\alpha\}$ and $\{\chi_\alpha\}$ is fermionic superfields required by the preservation of the superdimension and hence the total central charge. Under this dualization, the original complex superspace $\mathbb{C}^{p+n|f}$ gets mapped to

$$\mathbb{C}^{p+n+f|2f}. \tag{2.30}$$

The extended B-model, mirror to the above fermionic extended A-model with the superfield action $\mathcal{S}_{\mathcal{N}=2}$, is given in terms of the following path integral, see also [17]:

$$\mathcal{Z} = \int \mathcal{D}\mathcal{F} \left[{}_{a=1}^{p} \delta(\mathcal{F}_a - t_a)\right] \exp\left[\int \mathcal{W}(Y, X, \eta, \chi)\right], \tag{2.31}$$

where we have set $\mathcal{D}\mathcal{F} = \left(\prod_i \mathrm{d}Y_i\right)\left(\prod_\alpha \mathrm{d}X_\alpha\, \mathrm{d}\eta_\alpha\, \mathrm{d}\chi_\alpha\right)$. In this relation, the $\mathcal{F}_a$'s are the D-terms of the extended A-model and $\mathcal{W} = \mathcal{W}(Y, X, \eta, \chi)$ is the extended LG superpotential of the topological B-model. They are as follows:

$$\begin{aligned} \mathcal{F}_a &= \sum_{i=1}^{n+p} q_i^a Y_i - \sum_{\alpha=1}^{f} Q_\alpha^a X_\alpha, \qquad a = 1, \ldots, p, \\ \mathcal{W} &= \left(\sum_{i=1}^{n+p} \mathrm{e}^{-Y_i} + \sum_{\alpha=1}^{f} \mathrm{e}^{-X_\alpha}(1 + \eta_\alpha\chi_\alpha)\right). \end{aligned} \tag{2.32}$$

To extract information on the local supergeometry of the B-model, we need to integrate out the delta functions. Below, we shall focus our attention on the special case where $f = 2p$ and exemplify with models which have been used in type II superstring theory compactifications. This choice appears naturally in the study of ALE space with A_r supergeometries. In the case of $r = 1$, the minimal number of the fermionic fields, satisfying the local CY condition (2.22), is 2. We intend to address elsewhere the situation where $f \neq 2p$.

3. Mirror of A_r supergeometries

Here, we focus on the supergeometry extending the usual ordinary A_r geometries. A quite similar analysis is *a priori* possible for the DE, affine and indefinite extensions.

3.1. Local super A_1 geometry

To illustrate the construction, we initially consider the example of the model A_1. This is a supersymmetric gauge theory with a $U(1)$ gauge symmetry and three chiral superfields Φ_i with charge $(1, -2, 1)$ together with a real gauge superfield V. The D-term constraint (equation of motion of V) reads as

$$|\Phi_1|^2 - 2|\Phi_2|^2 + |\Phi_3|^2 = \mathrm{Re}(t). \tag{3.1}$$

This geometry describes the Kahler deformation of the A_1 singularity of the ALE spaces

$$uv = z^2, \tag{3.2}$$

where u, v and z are the generators of gauge invariants. They are realized in terms of the scalar fields as follows:

$$u = \Phi_1^2\Phi_2, \qquad v = \Phi_3^2\Phi_2, \qquad z = \Phi_1\Phi_2\Phi_3. \tag{3.3}$$

For generalizations to rank $r \geqslant 2$ ordinary ADE geometries as well as affine extensions and beyond see [23].

3.1.1. Extended model. Basically, there is an abundance of possible fermionic extensions of the above model. It may be limited by imposing the LSCY condition (2.21). Since we are interested in the case $f = 2p = 2$, the full spectrum of $U(1)$ charge that one can have is given by the vector

$$q' = (q|Q) = (1, -2, 1|1, -1). \tag{3.4}$$

In this construction, A_1 model appears as a sub-system while, as noted before and as far as the super-CY condition is concerned, there are several solutions of equation (2.21). Using extension (3.4), the D-term for the A_1 supergeometry becomes

$$|\Phi_1|^2 - 2|\Phi_2|^2 + |\Phi_3|^2 + |\Psi_1|^2 - |\Psi_2|^2 = \mathrm{Re}(t), \tag{3.5}$$

where $\mathrm{Re}(t)$ is the unique Kahler parameter of the model.

3.1.2. Mirror of the extended model. Applying mirror transformation to the above extended A-model with A_1 supergeometry, the associated mirror B-model is obtained in the same way as presented in the previous subsection. The corresponding extended LG path integral equations (2.31, 2.32) takes the following form:

$$\mathcal{Z} = \int \mathcal{DF}\,\delta[Y_1 - 2Y_2 + Y_3 - X_1 + X_2 - t]\exp\left(\sum_{i=1}^{3} \mathrm{e}^{-Y_i} + \sum_{\alpha=1}^{2} \mathrm{e}^{-X_\alpha}(1 + \eta_\alpha\chi_\alpha)\right), \tag{3.6}$$

with $\mathcal{DF} = \left(\prod_{i=1}^{3} \mathrm{d}Y_i\right)\left(\prod_{\alpha=1}^{2} \mathrm{d}X_\alpha\,\mathrm{d}\eta_\alpha\,\mathrm{d}\chi_\alpha\right)$. As usual, to extract information on the mirror supergeometry of the B-model, we integrate out the fermionic fields η_1, χ_1. Then, solving the delta function constraint by integrating out X_1 yields

$$\mathcal{Z} = \int \mathcal{D}\widetilde{\mathcal{F}}\,(\mathrm{e}^{-Y_1+2Y_2-Y_3}\,\mathrm{e}^{-X_2})\exp\left(\sum_{i=1}^{3} \mathrm{e}^{-Y_i} + \mathrm{e}^{-X_2}[1 + \eta_2\chi_2 + \mathrm{e}^{t}\,\mathrm{e}^{-Y_1+2Y_2-Y_3}]\right), \tag{3.7}$$

where $\mathcal{D}\widetilde{F} = \left(\prod_{i=1}^{3} \mathrm{d}Y_i\right)(\mathrm{d}X_2\, \mathrm{d}\eta_2\, \mathrm{d}\chi_2)$. Now, introducing the new complex variables x_i and y_i such that

$$x = \mathrm{e}^{-X_2}, \qquad y_i = \mathrm{e}^{-Y_i}, \qquad i = 1, 2, 3, \tag{3.8}$$

the above partition function becomes

$$\mathcal{Z} = \int (\mathrm{d}x\, \mathrm{d}\eta_2\, \mathrm{d}\chi_2) \prod_{i=1}^{3} \left(\frac{\mathrm{d}y_i}{y_2^3}\right) \exp\left(\sum_{i=1}^{3} y_i + x\left[1 + \eta_2\chi_2 + \mathrm{e}^t \frac{y_1 y_3}{y_2^2}\right]\right). \tag{3.9}$$

The rescaling $\tilde{x} = \left(x/y_2^3\right)$ allows us to rewrite the above path integral as follows:

$$\mathcal{Z} = \int \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}y_3\, \mathrm{d}\tilde{x}\, \mathrm{d}\eta_2\, \mathrm{d}\chi_2 \exp\left(\sum_{i=1}^{3} y_i + \tilde{x} y_2^3\left[1 + \eta_2\chi_2 + \mathrm{e}^t \frac{y_1 y_3}{y_2^2}\right]\right). \tag{3.10}$$

In order to get the mirror of the local supergeometry A_1, we can see $\tilde{x}$ as a Lagrange multiplier; integrating it out one gets the following equation of motion:

$$1 + \eta_2\chi_2 + \frac{y_1 y_3}{y_2^2}\, \mathrm{e}^t = 0. \tag{3.11}$$

The objective now is to interpret this equation as the mirror constraint equation of the topological A-model on A_1 supergeometry. In fact, we can solve (3.11) as

$$\frac{y_1 y_3}{y_2^2} = -(1 + \eta_2\chi_2)\, \mathrm{e}^{-t}. \tag{3.12}$$

Replacing now t by $t' = t + \mathrm{i}\pi$, one absorbs the minus sign

$$\frac{y_1 y_3}{y_2^2} = \mathrm{e}^{-t'} + \eta_2\chi_2\, \mathrm{e}^{-t'}. \tag{3.13}$$

Actually, this equation is quite similar to the bosonic one except that now we have the presence of the additional contribution $\eta_2\chi_2\, \mathrm{e}^{-t'}$, induced by the fermionic fields. It is easy to see that in the patch $\eta_2 = \chi_2 = 0$, we recover the bosonic mirror constraint equation of the ALE space with A_1 singularity, namely

$$\frac{y_1 y_3}{y_2^2} = \mathrm{e}^{-t'}. \tag{3.14}$$

Return to equation (3.13); a straightforward computation reveals that this equation can be solved by taking the following parameterization:

$$y_1 = y, \qquad y_3 = \frac{1}{y}, \qquad y_2 = (1 + \eta_2\chi_2)^{-\frac{1}{2}}\, \mathrm{e}^{\frac{t}{2}}, \tag{3.15}$$

where we have set $t' = t$. We thus end with the following LG potential:

$$y + \frac{1}{y} + (1 + \eta_2\chi_2)^{-\frac{1}{2}}\, \mathrm{e}^{\frac{t}{2}} = 0, \tag{3.16}$$

which is a mirror to the sigma model on A_1 supergeometry. This equation has three following remarkable features:

(1) For $\eta_2 = \chi_2$, we recover the usual bosonic LG superpotential mirror to the bosonic A_1 geometry

$$y + \frac{1}{y} + \mathrm{e}^{\frac{t}{2}} = 0. \tag{3.17}$$

(2) In the case $t \to 0$, one discovers the rule to define the superextension of the A_1 singularity with $U(1)$ charges as in equation (3.4). The mirror of the super A_1 singularity can then be defined as follows:

$$y_1 y_3 = y_2^2(1 + \eta_2\chi_2), \tag{3.18}$$

in agreement with an indication from conformal LG field models where adjunction of quadratic terms does not modify the total central charge. Moreover, by using fermionic statistics which forbids higher powers in η_2 and χ_2, one may define extensions of the above A_1 singularity.

(3) In the limit where the condensate modulus $\eta_2\chi_2$ is small, equation (3.16) reduces to

$$y + \frac{1}{y} + \mathrm{e}^{\frac{t}{2}} - \frac{1}{2}\eta_2\chi_2\,\mathrm{e}^{\frac{t}{2}} = 0. \tag{3.19}$$

By making the identification $\frac{1}{2}\eta_2\chi_2\,\mathrm{e}^{-\frac{t}{2}}$ with the uv of the relation (2.11–2.12), one discovers that the uv term added by hand in the bosonic case to recover the right dimension of the mirror manifold is generated in a natural way in supergeometry. In the limit $t \to 0$, we have

$$y + \frac{1}{y} + 1 = \eta_2'\chi_2', \tag{3.20}$$

where we have set $\eta_2'\chi_2' = \frac{1}{2}\eta_2\chi_2$. This is a complex two dimensions even hypersurface of $\mathbb{C}^{1|2}$.

3.2. *Super A_p*

Now, we would like to push further the above results on super A_1 to the class of A_p supergeometry series having usual A_p geometry as local bosonic Calabi–Yau sub-manifolds. To start with, recall that A_p geometry has a description in terms of the $U(1)^{\otimes p}$ sigma model involving $(p+2)$ chiral fields with the bosonic charge $p \times (p+2)$ matrix:

$$q_i^a = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -2 & 1 \end{pmatrix}. \tag{3.21}$$

Basically, there are several fermionic extensions of the above A-model. However as we mentioned before, we consider a model with $2p$ fermionic fields. In this way, the SLCY condition may limit the choice of the charge matrix. For a reason to be specified later on, we propose the following $U(1)^{\otimes p}$ charge spectrum for ghost-like superfields:

$$Q_\alpha^a = \begin{pmatrix} 1 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{3.22}$$

This representation constitutes a simple and natural extension of equation (3.4) recovering A_1 supergeometry as a leading example; other representations are obviously possible. This choice of $U(1)^{\otimes p}$ charge matrix for ghost-like fields allows us to handle each line as an individual

A_1 supergeometry. In this way, we can easily repeat the same lines that we have done for the super A_1 case. Let us give some details below.

Roughly, the LG mirror superpotential is given in terms of the following path integral:

$$\mathcal{Z} = \int \mathcal{D}F \left[_{a=1}^{p} \delta(F_a - t_a)\right] \exp\left(\sum_i \mathrm{e}^{-Y_i} + \sum_\alpha \mathrm{e}^{-X_\alpha}(1 + \eta_\alpha \chi_\alpha)\right), \qquad (3.23)$$

where now $\mathcal{D}F = \left(\prod_{i=1}^{p+2} \mathrm{d}Y_i\right)\left(\prod_{\alpha=1}^{2p} \mathrm{d}X_\alpha\, \mathrm{d}\eta_\alpha\, \mathrm{d}\chi_\alpha\right)$ and where we have set

$$F_a = Y_a - 2Y_{a+1} + Y_{a+2} - X_{2a-1} + X_{2a}. \qquad (3.24)$$

This partition function $\mathcal{Z}$ has p delta functions $\delta(F_a - t_a)$. To get the mirror supergeometry, we first integrate out the fermionic field variables $(\eta_{1a}\chi_{1a})$ leaving only a dependence on $(\eta_{2a}\chi_{2a})$, $a = 1, 2, \ldots, p$, and then we use delta functions to eliminate the field variables X_{2a-1}. In doing so and following the same way as before, we get p equations of motion,

$$1 + \eta_{2a}\chi_{2a} = \prod_i y_i^{q_i^a}, \qquad a = 1, \ldots, p. \qquad (3.25)$$

To see how to obtain these equations, let us consider the case of A_2 supergeometry. This is a $U(1)^2$ gauge theory with four chiral superfields $(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ and four ghost-like ones $(\Psi_1, \Psi_2, \Psi_3, \Psi_4)$. The full spectrum of $U(1)^2$ gauge charges is given by

$$q'^1 = (1, -2, 1, 0|1, -1, 0, 0), \qquad q'^2 = (0, 1, -2, 1|0, 0, 1, -1). \qquad (3.26)$$

The above path integral reduces, in the present case, to

$$\mathcal{Z} = \int \mathcal{D}F\, \delta(F_1 - t_1)\delta(F_2 - t_2) \exp\left(\sum_{i=1}^{4} \mathrm{e}^{-Y_i} + \sum_{\alpha=1}^{4} \mathrm{e}^{-X_\alpha}(1 + \eta_\alpha\chi_\alpha)\right), \qquad (3.27)$$

with field measure $\mathcal{D}F = \left(\prod_{i=1}^{4} \mathrm{d}Y_i\right)\left(\prod_{\alpha=1}^{4} \mathrm{d}X_\alpha\, \mathrm{d}\eta_\alpha\, \mathrm{d}\chi_\alpha\right)$ and D-terms as

$$F_1 = Y_1 - 2Y_2 + Y_3 - X_1 + X_2, \qquad F_2 = Y_2 - 2Y_3 + Y_4 - X_3 + X_4. \qquad (3.28)$$

Integrating in a similar way as we have done for A_1 supergeometry and making the same variable changes, we get

$$\mathcal{Z} = \int \mathcal{D}F' \exp\left[\sum_{i=1}^{4} y_i + \tilde{x}_1 y_2^2\left(1 + \eta_2\chi_2 + \frac{\mathrm{e}^{t_1} y_1 y_3}{y_2^2}\right)\right] \exp\left[\tilde{x}_2 y_3^2\left(1 + \eta_4\chi_4 + \frac{\mathrm{e}^{t_2} y_2 y_4}{y_3^2}\right)\right], \qquad (3.29)$$

with $\mathcal{D}F' = \left(\prod_{i=1}^{4} \mathrm{d}y_i\right)(\mathrm{d}\tilde{x}_1\, \mathrm{d}\tilde{x}_2\, \mathrm{d}\eta_2\, \mathrm{d}\chi_2\, \mathrm{d}\eta_4\, \mathrm{d}\chi_4)$. In this case, we have two equations of motion which are given by

$$\frac{y_1 y_3}{y_2^2} = (1 + \eta_2\chi_2)\,\mathrm{e}^{-t_1'}, \qquad \frac{y_2 y_4}{y_3^2} = (1 + \eta_4\chi_4)\,\mathrm{e}^{-t_2'}, \qquad (3.30)$$

with $t_a' = t_a + \mathrm{i}\pi$. After solving these two equations, we come up with the following mirror relation:

$$\frac{1}{y} + \left(\mathrm{e}^{\frac{t_1'}{2}}(1 + \eta_2\chi_2)\right) + y + y^2\left(\mathrm{e}^{-t_2'}\,\mathrm{e}^{\frac{-t_1'}{2}}(1 + \eta_4\chi_4)(1 + \eta_2\chi_2)^{\frac{1}{2}}\right) = 0, \qquad (3.31)$$

which should be compared with the usual mirror relation of ordinary A_2 geometry $\frac{1}{y} + 1 + y + y^2 = 0$ associated with the limit $t_a' = 0$ and $\eta_2 = \chi_2 = 0$.

4. More on mirror supergeometry

The method developed so far can also be used to build other local super-CY manifolds. A simple extension of the above A_1 supergeometry analysis is given by a sigma model with the target space involving a fermionic extension of a line bundle over $\mathbf{CP}^p$ with $p \geqslant 2$. The case $p = 1$ corresponds exactly to the A_1 supergeometry studied previously. Let us analyse the case $p = 2$, that is, the line bundle $\mathcal{O}(-3)$ over $\mathbf{CP}^2$. It admits a $U(1)$ sigma model description in terms of four bosonic chiral fields with charge vector $(1, 1, 1, -3)$, and the corresponding D-term equation is given by

$$|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 - 3|\Phi_4|^2 = \mathrm{Re}(t). \tag{4.1}$$

Adding now two ghost-like field variables Ψ_1 and Ψ_2 with vector charge $(1, -1)$, as required by the SLCY condition, the D-term constraint equation of the extended A-model is given by

$$|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 - 3|\Phi_4|^2 + |\Psi_1|^2 - |\Psi_2|^2 = \mathrm{Re}(t). \tag{4.2}$$

The corresponding mirror supergeometry is given in terms of the following path integral:

$$\mathcal{Z} = \int \mathcal{D}\mathcal{F}\, \delta\,(\mathcal{F} - t) \exp\left(\sum_{i=1}^{4} \mathrm{e}^{-Y_i} + \sum_{\alpha=1}^{2} \mathrm{e}^{-X_\alpha}(1 + \eta_\alpha \chi_\alpha)\right) \tag{4.3}$$

with $\mathcal{D}\mathcal{F} = \left(\prod_{i=1}^4 \mathrm{d}Y_i\right)\left(\prod_{\alpha=1}^2 \mathrm{d}X_\alpha\, \mathrm{d}\eta_\alpha\, \mathrm{d}\chi_\alpha\right)$ and

$$\mathcal{F} = Y_1 + Y_2 + Y_3 - 3Y_4 - X_1 + X_2. \tag{4.4}$$

Now integrating out the fermionic fields η_1, χ_1 and solving the delta function constraint by eliminating X_1, we get

$$\mathcal{Z} = \int \mathcal{D}\mathcal{F}\, (\mathrm{e}^{-Y_1-Y_2-Y_3+3Y_4}\, \mathrm{e}^{-X_2}) \exp\left(\sum_{i=1}^{4} \mathrm{e}^{-Y_i} + \mathrm{e}^{-X_2}(1 + \eta_2\chi_2 + \mathrm{e}^t\, \mathrm{e}^{-Y_1-Y_2-Y_3+3Y_4})\right), \tag{4.5}$$

with $\mathcal{D}\mathcal{F} = \left(\prod_{i=1}^4 \mathrm{d}Y_i\right)(\mathrm{d}X_2\, \mathrm{d}\eta_2\, \mathrm{d}\chi_2)$. Using the following field re-definition:

$$y_i = \mathrm{e}^{-Y_i}, \qquad x = \mathrm{e}^{-X_2} \tag{4.6}$$

the above equation becomes

$$\mathcal{Z} = \int \left(\prod_{i=1}^{4} \frac{\mathrm{d}y_i}{y_4^4}\right) (\mathrm{d}x\, \mathrm{d}\eta_2\, \mathrm{d}\chi_2) \exp\left[\sum_{i=1}^{4} y_i + x\left(1 + \eta_2\chi_2 + \frac{\mathrm{e}^t y_1 y_2 y_3}{y_4^3}\right)\right]. \tag{4.7}$$

With the help of the following rescaling $\tilde{x} = \frac{x}{y_4^4}$, the mirror geometry becomes

$$\mathcal{Z} = \int \mathrm{d}\tilde{x}\, \mathrm{d}\eta_2\, \mathrm{d}\chi_2 \prod_{i=1}^{4} \mathrm{d}y_i \exp\left[\sum_{i=1}^{4} y_i + \tilde{x} y_4^4 \left(1 + \eta_2\chi_2 + \frac{\mathrm{e}^t y_1 y_2 y_3}{y_4^3}\right)\right]. \tag{4.8}$$

In this case, the equation of motion reads as

$$\frac{y_1 y_2 y_3}{y_4^3} = -(1 + \eta_2\chi_2)\, \mathrm{e}^{-t}. \tag{4.9}$$

Absorbing the minus sign by replacing t by $t + \mathrm{i}\pi$, the above equation becomes

$$\frac{y_1 y_2 y_3}{y_4^3} = \mathrm{e}^{-t'} + \eta_2\chi_2\, \mathrm{e}^{-t'}. \tag{4.10}$$

This can be easily solved by the following parameterization:

$$y_1 = x, \qquad y_2 = y, \qquad y_3 = \frac{1}{xy} \qquad y_4 = (1 + \eta_2\chi_2)^{\frac{-1}{3}}\, \mathrm{e}^{\frac{t'}{3}}. \tag{4.11}$$

The superpotential describing the mirror of the supergeometry reads as

$$x + y + \frac{1}{xy} + (1 + \eta_2\chi_2)^{\frac{-1}{3}}\, \mathrm{e}^{\frac{t'}{3}} = 0. \tag{4.12}$$

For $\eta_2\chi_2 = 0$, we rediscover the usual bosonic relation.

5. Conclusion

In this paper, we have studied mirror symmetry of A-model on Calabi–Yau supermanifolds constructed as fermionic extensions of local toric CY satisfying the SLCY condition $\sum_{i=1}^{p+n} q_i^a = \sum_{\alpha=1}^{2p} Q_\alpha^a$. By solving this condition as $\sum_{i=1}^{p+n} q_i^a = 0$ and $\sum_{\alpha=1}^{2p} Q_\alpha^a = 0$ separately, we have considered two classes of mirror supergeometries. The first class deals with a special fermionic extension of ordinary geometries and the second class concerns a set of sigma models with the target space involving a fermionic extension of a line bundle over $\mathbf{CP}^n$ with $n \geqslant 2$. The representations studied here are not the general ones since the bosonic sub-variety of a supermanifold considered here is taken as a Calabi–Yau manifold. This condition is obviously not a necessary condition for building Calabi–Yau supermanifolds. This work may be viewed as a an extension of [17] which has dealt with bosonic compact toric varieties. The mirror geometries studied in that paper have dealt only with bosonic variables. However, here the mirror B-models involve fermionic directions captured by the ghost-like fields. In dealing with the mirror of A_r supergeometries, we have shown that these local CY supermanifolds are described by algebraic geometry equations quite similar to the bosonic case. The later can be obtained by cancelling fermionic directions. Moreover, we have found that in supergeometry, the right dimension of the bosonic CY sub-variety is recovered in a natural way as shown in (3.20). Finally, we have shown that this approach applies as well to higher dimensional mirror supergeometries; the mirror of the A-model on the superline bundle over $\mathbf{CP}^n$ studied in section 4 is an example amongst others.

In the end of this study, we would like to add that along with ordinary CY manifolds embedded in $\mathbb{C}^{n|0}$ and super-CY manifolds embedded in complex space $\mathbb{C}^{n|m}$, we may also have super-CY manifolds embedded in the purely fermionic space $\mathbb{C}^{0|m}$ without basic bosonic coordinates. These special super-CY varieties are then hypersurfaces in $\mathbb{C}^{0|m}$ involving ghost-like fields only.

Our work opens up for further studies. A natural extension of the present work includes the case $p \neq 2f$. Another interesting problem is to give the general solution for the mirror super-CY manifolds. This may generalize [18]. We hope to report elsewhere on these open questions.

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